Efficiency change over time in a multisectoral economic system

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“Efficiency and Productivity Analysis of Multisectoral Economic Systems”  
(supported by the Jubiläumsfonds of the Oesterreichischen Nationalbank)

Ottawa, 6 June 2014
Recessions are easily recognizable from a decrease in GDP. What really should interest us, however, is the difference between the potential of an economy and its actual performance. (J. Stiglitz, 2002)
Structure of the talk

- Motivation
- Leontief’s input-output model
- Production possibility set of an economy
- Relationship between DEA model and LP-Leontief model
- Productivity change of the economy over time
- Empirical application
- Conclusions and outlook to further research
Motivation (1)

Two approaches of productivity and efficiency analysis:

- **Neoclassical** approach
  
  ... *weights inputs by value shares* (requires data on factor input shares or prices)

  ... imputes productivity growth to factors, but cannot distinguish a movement towards the efficiency frontier and a movement of the latter

- **Frontier** approach
  
  ... allows *decomposing productivity growth into a movement of the economy towards the frontier and a shift of the latter*

  ... cannot impute productivity growth to factors
Motivation (2)

Bridges between these two approaches:

- **Ten Raa and Mohnen (2002)**
  
  ... estimated total factor productivity (TFP) growth without recourse to data on factor input prices

  ... reproduced the neoclassical TFP growth formulas, but in a framework that is Data Envelopment Analysis (DEA) in spirit

- **Luptacik and Böhm (2010)**

  ... represents the economy by the Leontief input-output model extended by the constraints for primary factors

  ... the efficiency frontier of the economy is generated by using the multi-objective optimization model

  ... the efficiency of the economy can be obtained as a solution of a DEA model with the virtual DMUs
Research Questions

Productivity change:

• How big is it?

• Where does it come from? Efficiency change or technical change?

• What are the main drivers? Is it output growth or input-saving?

• How much do individual outputs (agriculture, manufacturing, services, ...) and primary factors (capital, labor, ...) contribute?
The procedure consists of three steps:

1. **Generate the production possibility set**: Each output is maximised subject to restraints on the production of other outputs and available inputs (multi-objective optimisation problem).

2. **Measure distance** of actual economy to the production frontier.

3. **Efficiency change, technology change and productivity change over time** based on Luenberger Indicator.
# Growth accounting vs. our approach

<table>
<thead>
<tr>
<th></th>
<th>growth accounting</th>
<th>Our approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregation of primary inputs</td>
<td>Value shares or market prices</td>
<td>Shadow prices</td>
</tr>
<tr>
<td>Number of inputs and number of outputs</td>
<td>Multiple inputs - one output (mostly value added)</td>
<td>Multiple inputs - multiple outputs</td>
</tr>
<tr>
<td>Competition</td>
<td>Perfect competition</td>
<td>Non-perfect competition</td>
</tr>
<tr>
<td>Substitutes vs. complements</td>
<td>Primary factors are substitutes</td>
<td>Primary factors are complements</td>
</tr>
<tr>
<td>Efficiency change vs. technology change</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Number of countries</td>
<td>Often multi-country models (sample of different countries)</td>
<td>Single country model</td>
</tr>
<tr>
<td>Interdependences between sectors</td>
<td>Does not account for</td>
<td>Accounts for</td>
</tr>
<tr>
<td>Returns to scale</td>
<td>Constant</td>
<td>Constant</td>
</tr>
</tbody>
</table>
Leontief’s input-output model (1)

Economy with **n sectors**; Each sector produces a single homogeneous good, $x_j$. The $j$-th sector, in order to produce 1 unit, must use $a_{kj}$ units from sector $k$. Furthermore, each sector sells some of its output to other sectors (intermediate output) and some of its output to consumers (net output, or final demand). Call final demand in the $j$-th sector $y_j$. Then we might write

$$x_{j; t} = a_{j1; t}x_{1; t} + a_{j2; t}x_{2; t} + ... + a_{jn; t}x_{n; t} + y_{j; t}$$

or total output equals intermediate demand plus final demand. If we let $A$ be the indecomposable matrix of input coefficients $a_{kj}$, $x$ be the vector of total (gross) output, and $y$ be the vector of final demand/net output, then our expression for the economy becomes

$$x_t = A_t x_t + y_t.$$ 

**For given final demand the gross output must at least cover the intermediate output and final demand** which can be written as

$$x_t \geq A_t x_t + y_t^0 \quad \text{or} \quad (I - A_t)x_t \geq y_t^0 \quad \text{(1a)}$$
Leontief’s input-output model (2)

The economy uses \textit{m primary factors}. Moreover, the \( j \)-th sector, in order to produce 1 unit, must use \( b_{ij} \) units of the \( i \)-th primary factor. Then we might write

\[
b_{i1:t}x_{1:t} + b_{i2:t}x_{2:t} + \ldots + b_{in:t}x_{n:t} = z_{i:t}
\]

where \( b_{ij} \) the requirement of the \( j \)-th sector on the \( i \)-th primary factor and \( z_i \) the endowment of the \( i \)-th primary factor. Let \( B \) be the matrix of primary factor coefficients \( b_{ij} \) and \( z \) be the vector of total factor endowments. Then \textbf{the sum of primary factors used by all sectors can not exceed the total endowments in the economy:}

\[
B_t x_t \leq z_{t}^0
\]
Production possibility set of the economy (1)

What is the maximum possible net output/final demand given the endowment of primary factors?

Each net output $y$ is maximized s.t. restrictions on availability of inputs $z^0$:

$$\text{Max } y_t \quad \text{s.t.}$$

$$(I - A_t)x_t - y_t \geq 0$$

$$(I - A_t)x_t \quad \text{s.t.}$$

$$B_t x_t \leq z^0_t$$

$$B_t x_t - z_t \leq 0$$

$x_t, y_t \geq 0$

What is the minimum primary factor required to satisfy the given level of final demand?

For given level of final demand $y^0$ the use of inputs $z$ is minimized:

$$\text{Min } z_t \quad \text{s.t.}$$

$$(I - A_t)x_t \geq y^0$$

$$(I - A_t)x_t \quad \text{s.t.}$$

$$B_t x_t \leq z^0_t$$

$$B_t x_t - z_t \leq 0$$

$x_t, z_t \geq 0$
Instead of the multi-objective model we solve $n$ single-objective:

$$\max y_{j;t} \ (j = 1, \ldots, n)$$ (4)

subject to the constraints in (2).

The solution vector $y_{t}^{*j} (j = 1, \ldots, n)$ represents the net-output.

Instead of the multi-objective model we solve $m$ single-objective:

$$\min z_{i;t} \ (i = 1, \ldots, m)$$ (5)

subject to the constraints in (3).

The solution vector $z_{t}^{*i} (i = 1, \ldots, m)$ denotes the optimal input.

Both sets of solutions will be inserted in the following pay-off matrix:

$$P_{t,t} = \begin{bmatrix}
  y_{t}^{*1} & y_{t}^{*2} & \ldots & y_{t}^{*n} \\
  z_{t}^{0} - s_{z}^{1} & z_{t}^{0} - s_{z}^{2} & \ldots & z_{t}^{0} - s_{z}^{n}
\end{bmatrix}
\begin{bmatrix}
  y_{t}^{0} + s_{y}^{1} & y_{t}^{0} + s_{y}^{2} & \ldots & y_{t}^{0} + s_{y}^{m} \\
  z_{t}^{*1} & z_{t}^{*2} & \ldots & z_{t}^{*m}
\end{bmatrix}
\equiv
\begin{bmatrix}
P_{1;t,t} \\
P_{2;t,t}
\end{bmatrix}$$

where $s_{y}$ is the vector of the slack variables of the $n$ outputs and $s_{z}$ is the vector of the $m$ input slacks.
P is used to **establish the frontier of the production possibility set** (or the input requirement set) i.e. the efficient envelope.

This efficient envelope is used to **evaluate the relative (in-) efficiency of the economy** given the actual output and input data \((y^0, z^0)\) in the following non-oriented DEA model:

\[
\rho_t(z_t^0, y_t^0) = \max_{\mu, \beta} \beta \quad \text{s.t.} \\
- \beta y_t^0 + P_{1; t, t} \mu \geq y_t^0 \\
\beta z_t^0 + P_{2; t, t} \mu \leq z_t^0 \\
\mu \geq 0, \beta \text{ free}
\]  

(6)
The relationship between the DEA model and the LP-Leontief model

In the spirit of ten Raa (1995, 2005) and Debreu (1951) the Leontief-model can be formulated as an optimization problem in the following way:

\[
\omega_t(z_t^0, y_t^0) = \max_{x, \delta} \delta \quad \text{s.t.}
\]

\[
-\delta y_t^0 + (I - A_t)x_t \geq y_t^0
\]

\[
\delta z_t^0 + B_t x_t \leq z_t^0
\]

\[
x_t \geq 0, \delta \text{ free}
\]

(7)

**Proposition 1:** The efficiency score \( \rho \) of DEA problem (6) is exactly equal to the efficiency measure \( \omega \) of LP-model (7). The dual solution of model (7) coincides with the solution of the DEA multiplier problem which is the dual of problem (6).
Productivity change of the economy over time (1)

The procedure shown above can be applied for inter-temporal analysis. For this purpose the well known **Luenberger-indicator** can be adopted.

\[ L\left(z_t^0, y_t^0, z_{t+1}^0, y_{t+1}^0\right) = \frac{1}{2} \left[ \left( \rho_{t+1}(z_t^0, y_t^0) - \rho_{t+1}(z_{t+1}^0, y_{t+1}^0) \right) \right. \]

\[ + \left. \left( \rho_t(z_t^0, y_t^0) - \rho_t(z_{t+1}^0, y_{t+1}^0) \right) \right] \]

where subscript \( t \) denotes time period and \( \rho \) distance functions.

The four distance function values (two single period for \( t \) and \( t+1 \) and two mixed-period distance functions) can be estimated by solving the **DEA model (6)** for the respective time period. For each DEA model a separate output matrix \( P_1 \) and a separate input matrix \( P_2 \) have to be constructed by solving the LPs (4) and (5). Consequently, these two models as well as model (6) have to be used four times.
Productivity change of the economy over time (2)

Productivity change obtained from the Luenberger indicator can be decomposed into a component of efficiency change (catch-up) and technology change (frontier shift), like for any other Luenberger indicators.

Efficiency change:  
\[
EFFCH\left(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}\right) = \rho_t\left(z^0_t, y^0_t\right) - \rho_{t+1}\left(z^0_{t+1}, y^0_{t+1}\right)
\]

Technology change:  
\[
TECHCH\left(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}\right) = \frac{1}{2}\left[\left(\rho_{t+1}\left(z^0_{t+1}, y^0_{t+1}\right) - \rho_t\left(z^0_{t+1}, y^0_t\right)\right) - \rho_t\left(z^0_{t+1}, y^0_{t+1}\right)\right]
\]

Productivity change:
\[
L\left(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}\right) = EFFCH\left(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}\right) + TECHCH\left(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}\right)
\]
Productivity change of the economy over time (3)

The proposed method attributes the use of individual inputs and the final demand of individual commodities to productivity change and its components.

Efficiency change:

\[
EFFCH(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}) = \rho_t(z^0_t, y^0_t) - \rho_{t+1}(z^0_{t+1}, y^0_{t+1}) = \\
= \sum_{j=1}^{n} u_{j; t} y^0_{j; t} - \sum_{j=1}^{n} u_{j; t+1, t+1} y^0_{j; t+1} + \sum_{i=1}^{m} v_{i; t} z^0_{i; t} - \sum_{i=1}^{m} v_{i; t+1, t+1} z^0_{i; t+1}
\]

Contribution of the \(i\)-th input:

\[
v_{i; t} z^0_{i; t} - v_{i; t+1, t+1} z^0_{i; t+1}
\]

Contribution of the \(j\)-th output:

\[
u_{j; t} y^0_{j; t} - u_{j; t+1, t+1} y^0_{j; t+1}
\]
Productivity change of the economy over time (4)

Technical change:

\[\text{TECHCH}(z^0_t, y^0_t, z^0_{t+1}, y^0_{t+1}) = \frac{1}{2} \left[ \rho_{t+1}(z^0_{t+1}, y^0_{t+1}) - \rho_t(z^0_t, y^0_t) + \rho_t(z^0_{t+1}, y^0_{t+1}) - \rho_{t+1}(z^0_t, y^0_t) \right] = \frac{1}{2} \left[ \sum_{j=1}^{n} u_{j:t},_{t+1} y^0_{j:t+1} - \sum_{j=1}^{n} u_{j:t},_{t+1} y^0_{j:t+1} + \sum_{j=1}^{n} u_{j:t},_{t} y^0_{j:t} - \sum_{j=1}^{n} u_{j:t},_{t} y^0_{j:t} - \sum_{i=1}^{m} v_{i:t},_{t+1} z^0_{i:t+1} + \sum_{i=1}^{m} v_{i:t},_{t+1} z^0_{i:t+1} + \sum_{i=1}^{m} v_{i:t+1},_{t} z^0_{i:t} - \sum_{i=1}^{m} v_{i:t+1},_{t} z^0_{i:t} - \sum_{i=1}^{m} v_{i:t},_{t} z^0_{i:t} \right] \]

Contribution of the \(i\)-th input:

\[\frac{1}{2} \left( v_{i:t+1},_{t+1} z^0_{i:t+1} - v_{i:t},_{t+1} z^0_{i:t+1} + v_{i:t+1},_{t} z^0_{i:t} - v_{i:t},_{t} z^0_{i:t} \right)\]

Contribution of the \(j\)-th output:

\[\frac{1}{2} \left( u_{j:t+1},_{t+1} y^0_{j:t+1} - u_{j:t},_{t+1} y^0_{j:t+1} + u_{j:t+1},_{t} y^0_{j:t} - u_{j:t},_{t} y^0_{j:t} \right)\]
Productivity change of the economy over time (5)

Productivity change:

\[ L(z_t^0, y_t^0, z_{t+1}^0, y_{t+1}^0) = \frac{1}{2} \left[ (\rho_{t+1}(z_t^0, y_t^0) - \rho_{t+1}(z_{t+1}^0, y_{t+1}^0)) + (\rho_t(z_t^0, y_t^0) - \rho_t(z_{t+1}^0, y_{t+1}^0)) \right] = \]

\[ = \frac{1}{2} \left[ \sum_{j=1}^{n} u_{j+1,t} y_0^{j,t} - \sum_{j=1}^{n} u_{j+1,t+1} y_0^{j,t+1} + \sum_{j=1}^{n} u_{j,t} y_0^{j,t} - \sum_{j=1}^{n} u_{j,t+1} y_0^{j,t+1} - \right. \]

\[ \left. + \sum_{i=1}^{m} v_{i+1,t} z_0^{i,t} - \sum_{i=1}^{m} v_{i+1,t+1} z_0^{i,t+1} + \sum_{i=1}^{m} v_{i,t} z_0^{i,t} - \sum_{i=1}^{m} v_{i,t+1} z_0^{i,t+1} \right] \]

Contribution of the \( i \)-th input:

\[ \frac{1}{2} \left( v_{i+1,t} z_0^{i,t} - v_{i+1,t+1} z_0^{i,t+1} + v_{i,t} z_0^{i,t} - v_{i,t+1} z_0^{i,t+1} \right) \]

Contribution of the \( j \)-th output:

\[ \frac{1}{2} \left( u_{j+1,t} y_0^{j,t} - u_{j+1,t+1} y_0^{j,t+1} + u_{j,t} y_0^{j,t} - u_{j,t+1} y_0^{j,t+1} \right) \]
Empirical Application (1)

**Country:** US economy

**Observation period:** 1977 to 2006

**Input-Output Tables:** aggregated to 6 industry / commodity sectors, based on domestic use tables

**Final demand:** 6 commodities

**Primary factors:** High-skilled labour, Medium-skilled labour, Low-skilled labour, Capital stock, all assets

**Data sources:** Miller and Blair (2010) and EU KLEMS Database (2011)
Descriptive statistics of primary factors:

<table>
<thead>
<tr>
<th></th>
<th>used</th>
<th>endowment</th>
<th>ratio used to endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in 1977</td>
<td></td>
<td></td>
</tr>
<tr>
<td>High-skilled labor</td>
<td>32,021</td>
<td>38,353</td>
<td>0.83</td>
</tr>
<tr>
<td>(in Mill. hours)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medium-skilled labor</td>
<td>101,139</td>
<td>220,071</td>
<td>0.46</td>
</tr>
<tr>
<td>(in Mill. hours)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low-skilled labor</td>
<td>41,098</td>
<td>52,607</td>
<td>0.78</td>
</tr>
<tr>
<td>(in Mill. hours)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital, all assets</td>
<td>12,949</td>
<td>15,530</td>
<td>0.83</td>
</tr>
<tr>
<td>(in Bill. USD)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>in 2006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>High-skilled labor</td>
<td>80,086</td>
<td>100,773</td>
<td>0.79</td>
</tr>
<tr>
<td>(in Mill. hours)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medium-skilled labor</td>
<td>147,965</td>
<td>307,790</td>
<td>0.48</td>
</tr>
<tr>
<td>(in Mill. hours)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low-skilled labor</td>
<td>24,934</td>
<td>36,030</td>
<td>0.69</td>
</tr>
<tr>
<td>(in Bill. USD)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital, all assets</td>
<td>29,278</td>
<td>36,429</td>
<td>0.80</td>
</tr>
<tr>
<td>(in Bill. USD)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Empirical Application (3)

Descriptive statistics of **final demand**:

<table>
<thead>
<tr>
<th></th>
<th>1977</th>
<th>2006</th>
<th>growth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in Bill. USD</td>
<td>in percent</td>
<td></td>
</tr>
<tr>
<td>Agriculture</td>
<td>43.7</td>
<td>47.7</td>
<td>9.23</td>
</tr>
<tr>
<td>Construction</td>
<td>755.8</td>
<td>1,219.9</td>
<td>61.40</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>1,214.7</td>
<td>1,691.0</td>
<td>39.21</td>
</tr>
<tr>
<td>Trade, Transport. &amp; Utilities</td>
<td>821.1</td>
<td>2,129.8</td>
<td>159.40</td>
</tr>
<tr>
<td>Services</td>
<td>2,045.6</td>
<td>6,053.6</td>
<td>195.93</td>
</tr>
<tr>
<td>Others</td>
<td>581.6</td>
<td>2,063.3</td>
<td>254.79</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>5,462.5</strong></td>
<td><strong>13,205.4</strong></td>
<td><strong>141.75</strong></td>
</tr>
</tbody>
</table>
Matrix of primary factor requirement (B-matrix):

**1977**

<table>
<thead>
<tr>
<th></th>
<th>Agriculture</th>
<th>Construction</th>
<th>Manufacturing</th>
<th>Trade, Transport &amp; Utilities</th>
<th>Services</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-skilled Labour</td>
<td>3.18</td>
<td>0.73</td>
<td>1.51</td>
<td>3.40</td>
<td>2.09</td>
<td>12.14</td>
</tr>
<tr>
<td>Medium-skilled Labour</td>
<td>20.86</td>
<td>5.61</td>
<td>7.65</td>
<td>18.18</td>
<td>5.44</td>
<td>21.77</td>
</tr>
<tr>
<td>Low-skilled Labour</td>
<td>19.12</td>
<td>3.22</td>
<td>4.30</td>
<td>6.04</td>
<td>1.47</td>
<td>7.02</td>
</tr>
<tr>
<td>capital total</td>
<td>1.95</td>
<td>0.11</td>
<td>0.37</td>
<td>1.24</td>
<td>2.34</td>
<td>1.79</td>
</tr>
</tbody>
</table>

**2006**

<table>
<thead>
<tr>
<th></th>
<th>Agriculture</th>
<th>Construction</th>
<th>Manufacturing</th>
<th>Trade, Transport &amp; Utilities</th>
<th>Services</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-skilled Labour</td>
<td>4.03</td>
<td>1.45</td>
<td>1.53</td>
<td>3.13</td>
<td>2.14</td>
<td>10.77</td>
</tr>
<tr>
<td>Medium-skilled Labour</td>
<td>14.72</td>
<td>8.76</td>
<td>3.78</td>
<td>10.07</td>
<td>2.93</td>
<td>13.82</td>
</tr>
<tr>
<td>Low-skilled Labour</td>
<td>5.79</td>
<td>2.73</td>
<td>0.75</td>
<td>1.55</td>
<td>0.49</td>
<td>1.42</td>
</tr>
<tr>
<td>capital total</td>
<td>1.27</td>
<td>0.13</td>
<td>0.33</td>
<td>0.82</td>
<td>1.58</td>
<td>1.96</td>
</tr>
</tbody>
</table>
Empirical Application (5)

Results for single period DEA (model 6) and Leontief model (model 7) for 1977 and 2006:

<table>
<thead>
<tr>
<th></th>
<th>In-efficiency score</th>
<th>shadow pr. high-sk. labor</th>
<th>shadow pr. med.-sk. labor</th>
<th>shadow pr. low-sk. labor</th>
<th>shadow pr. capital</th>
</tr>
</thead>
<tbody>
<tr>
<td>in 1977</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEA model</td>
<td>0.090</td>
<td>0.00001</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Leontief model</td>
<td>0.090</td>
<td>0.00001</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>in 2006</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEA model</td>
<td>0.109</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00002</td>
</tr>
<tr>
<td>Leontief model</td>
<td>0.109</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00002</td>
</tr>
</tbody>
</table>
Empirical Application (6)

Results for single period DEA (model 6) and Leontief model (model 7) for 1977 and 2006:

<table>
<thead>
<tr>
<th></th>
<th>shadow pr. Agriculture</th>
<th>shadow pr. Construction</th>
<th>shadow pr. Manufacturing</th>
<th>shadow pr. Trade &amp; Transportation</th>
<th>shadow pr. Services</th>
<th>shadow pr. Other</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>in 1977</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEA model</td>
<td>-0.00013</td>
<td>-0.00005</td>
<td>-0.00011</td>
<td>-0.00009</td>
<td>-0.00005</td>
<td>-0.00018</td>
</tr>
<tr>
<td>Leontief model</td>
<td>-0.00013</td>
<td>-0.00005</td>
<td>-0.00011</td>
<td>-0.00009</td>
<td>-0.00005</td>
<td>-0.00018</td>
</tr>
<tr>
<td><strong>in 2006</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEA model</td>
<td>-0.00004</td>
<td>-0.00002</td>
<td>-0.00003</td>
<td>-0.00003</td>
<td>-0.00004</td>
<td>-0.00004</td>
</tr>
<tr>
<td>Leontief model</td>
<td>-0.00004</td>
<td>-0.00002</td>
<td>-0.00003</td>
<td>-0.00003</td>
<td>-0.00004</td>
<td>-0.00004</td>
</tr>
</tbody>
</table>
Empirical Application (7)

Results of Luenberger indicator and components, 1977 to 2006:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>DEA model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.090</td>
<td>0.109</td>
<td>-0.418</td>
<td>0.093</td>
<td>-0.019</td>
<td>0.265</td>
<td>0.246</td>
</tr>
<tr>
<td>Leontief model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.090</td>
<td>0.109</td>
<td>-0.418</td>
<td>0.093</td>
<td>-0.019</td>
<td>0.265</td>
<td>0.246</td>
</tr>
</tbody>
</table>

Efficiency change = 0.090 - 0.109 = -0.019
## Empirical Application (8)

<table>
<thead>
<tr>
<th>Output</th>
<th>Efficiency change</th>
<th>Technical change</th>
<th>TFP change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>-0.004</td>
<td>0.006</td>
<td>0.002</td>
</tr>
<tr>
<td>Construction</td>
<td>-0.013</td>
<td>0.017</td>
<td>0.004</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>-0.086</td>
<td>0.075</td>
<td>-0.010</td>
</tr>
<tr>
<td>Trade, transport. &amp; utilities</td>
<td>-0.023</td>
<td>0.062</td>
<td>0.039</td>
</tr>
<tr>
<td>Services</td>
<td>0.133</td>
<td>-0.048</td>
<td>0.086</td>
</tr>
<tr>
<td>Others</td>
<td>-0.017</td>
<td>0.020</td>
<td>0.003</td>
</tr>
<tr>
<td>Input</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High-skilled labor</td>
<td>0.545</td>
<td>0.001</td>
<td>0.546</td>
</tr>
<tr>
<td>Medium-skilled labor</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Low-skilled labor</td>
<td>0</td>
<td>-0.146</td>
<td>-0.146</td>
</tr>
<tr>
<td>Capital</td>
<td>-0.554</td>
<td>0.277</td>
<td>-0.277</td>
</tr>
<tr>
<td>Total</td>
<td>-0.019</td>
<td>0.265</td>
<td>0.246</td>
</tr>
</tbody>
</table>
Empirical Application (9)

The diagram illustrates the technical change and efficiency change for various sectors:

- Agriculture: 0.08
- Construction: -0.02
- Manufacturing: 0.06
- Trade, transportation & utilities: 0.13
- Services: -0.05
- Others: -0.02
- High-skilled labour: 0.54
- Medium-skilled labour: 0.00
- Low-skilled labour: 0.00
- Capital: 0.28

The technical change is represented in dark blue, while the efficiency change is in light blue.
Conclusions

The construction of the efficiency frontier permits an assessment with respect to the own potential of an economy defined by the given technology (even in the case of multiple outputs and inputs) without the need to compare it with other economies possessing possibly different technologies and obvious mutual interdependencies due to international trade.

Due to our results the relative merits of both approaches (frontier approach and growth accounting) can be used.

For inter-temporal comparisons of productivity growth the movement of the economy towards the frontier and its shift can be obtained by using the DEA formulation.

Next step: enlarge the model by pollution (augmented Leontief model) and measure eco-efficiency change and eco-productivity change over time.
Appendix: proof of proposition (1)

**Proof**

The dual model to (7) can be written

\[
\begin{align*}
\min & \quad p' y^0 - r' z^0 \\
\text{s.t.} & \quad p'(I - A) - r' B \geq 0 \\
& \quad -r' z^0 \geq \frac{1}{m} \\
& \quad p \geq 0
\end{align*}
\]  

\[(8)\]

The dual model to (6) is

\[
\begin{align*}
\min & \quad u' y^0 - v' z^0 \\
\text{s.t.} & \quad u' P_1 - v' P_2 \geq 0 \\
& \quad -v' z^0 \geq \frac{1}{m} \\
& \quad u \geq 0
\end{align*}
\]

\[(9)\]

\(p \ldots \) shadow prices of the \(n\) commodities and  
\(r \ldots \) shadow prices of the \(m\) primary factors  
\(u \ldots \) shadow prices of the \(n\) commodities and  
\(v \ldots \) shadow prices of the \(m\) primary factors
Appendix: proof of proposition (2)

Because of the in decomposability of the matrix A the vector $x$ must be positive and from the complementary slackness theorem follows

$$p'(I - A) - r' B = 0 \quad \text{and} \quad p' = r' B (I - A)^{-1} > 0$$

Multiplying the Leontief inverse by the matrix of generated net outputs $P_1$ we obtain the corresponding total gross output requirements, denoted by matrix $T$:

$$(I - A)^{-1} P_1 = T \geq 0 \quad (10)$$

In other words $T$ represents the total output requirements for each virtual decision making unit. Consequently

$$B T = B (I - A)^{-1} P_1$$

gives the necessary amount of primary inputs to satisfy the generated total output requirements. This coincides with the construction of matrix $P_2$ describing the primary input requirements necessary to satisfy final demands $P_1$. Therefore

$$P_2 = B T \quad (11)$$
Appendix: proof of proposition (3)

It follows from (10) that

\[ P_1 = (I - A)T \quad (12) \]

Multiplying the first constraint in (8) by \( T \) yields

\[ p'(I - A)T - r'BT \leq 0 \quad (13) \]

Substituting (11) and (12) for \( P_2 \) and \( P_1 \) respectively into (13) we obtain exactly the constraints as of the dual problem (9):

\[ p'P_1 - r'P_2 \leq 0 \]

Since we have two problems with the same constraints we have \( r_i = v_i \). The coefficients of the objective functions are the same. Therefore the optimal values of the objective functions must be the same:

\[ p'y^0 - r'z^0 = u'y^0 - v'z^0 \]

Consequently \( p' = u' \) and according to the duality theorem of linear programming \( \sigma = \rho \).
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